

SCATTERING ABOVE ENERGY NORM OF SOLUTIONS OF A FOCUSING LOGLOGLOG ENERGY-SUPERCritical SCHRÖDINGER EQUATION WITH RADIAL DATA BELOW GROUND STATE

TRISTAN ROY

ABSTRACT. Given $n \in \{3, 4\}$ and $k > \frac{n}{2}$, we prove scattering of the radial $\tilde{H}^k := \dot{H}^k(\mathbb{R}^n) \cap \dot{H}^1(\mathbb{R}^n)$ - solutions of the logloglog focusing energy-supercritical Schrödinger equation $i\partial_t u + \Delta u = -|u|^{\frac{4}{n-2}} u \log^\gamma \log(\log(10^{10} + |u|^2))$ for a range of positive γ 's, for energies below that of the ground states, and for potentials below that of the ground states. The proof uses in particular arguments of [2, 9, 11, 13, 14].

1. INTRODUCTION

We shall study the radial solutions of the following Schrödinger equation in dimension n , $n \in \{3, 4\}$:

$$(1) \quad i\partial_t u + \Delta u = -|u|^{\frac{4}{n-2}} u g(|u|)$$

with $g(|u|) := \log^\gamma \log(\log(10^{10} + |u|^2))$ and $\gamma > 0$.

This equation has many connections with the following power-type Schrödinger equation, $p > 1$

$$(2) \quad i\partial_t v + \Delta v = -|v|^{p-1} v$$

(2) has a natural scaling: if v is a solution of (2) with data $v(0) := v_0$ and if $\lambda \in \mathbb{R}$ is a parameter then $v_\lambda(t, x) := \frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$ is also a solution of (2) but with data $v_\lambda(0, x) := \frac{1}{\lambda^{\frac{2}{p-1}}} u_0\left(\frac{x}{\lambda}\right)$. If $s_p := \frac{n}{2} - \frac{2}{p-1}$ then the \dot{H}^{s_p} norm of the initial data is invariant under the scaling: this is why (2) is said to be \dot{H}^{s_p} -critical. If $p = 1 + \frac{4}{n-2}$ then (2) is \dot{H}^1 (or energy) critical. The energy-critical Schrödinger equation

$$(3) \quad i\partial_t u + \Delta u = -|u|^{\frac{4}{n-2}} u$$

has received a great deal of attention. Cazenave and Weissler [3] proved the local well-posedness of (3): given any $u(0)$ such that $\|u(0)\|_{\dot{H}^1} < \infty$ there exists, for some t_0 close to zero, a unique $u \in \mathcal{C}([0, t_0], \dot{H}^1) \cap L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}([0, t_0])$ satisfying (3) in the sense of distributions

$$(4) \quad u(t) = e^{it\Delta} u(0) + i \int_0^t e^{i(t-t')\Delta} \left[|u(t')|^{\frac{4}{n-2}} u(t') \right] dt'.$$

The next step is to understand the asymptotic behavior of the solutions of (3). It is well-known that (3) has a family of stationary solutions (the ground states) $W_{\lambda,\theta}(x) := e^{i\theta} \frac{1}{\lambda^{\frac{n-2}{2}}} W\left(\frac{x}{\lambda}\right)$ that satisfy $\Delta W_{\lambda,\theta} + |W_{\lambda,\theta}|^{\frac{4}{n-2}} W_{\lambda,\theta} = 0$ with $\theta \in [0, 2\pi]$ and W defined by

$$W(x) := \frac{1}{\left(1 + \frac{|x|^2}{n(n-2)}\right)^{\frac{n-2}{2}}}.$$

The asymptotic behavior of the solutions for energies below that of the ground states has been studied in [7]. In particular global existence and scattering (i.e the linear asymptotic behavior) were proved for potential energies below that of the ground states. The asymptotic behavior of the solutions was studied in [5] for energies equal to that of the ground states and in [9] for energies slightly larger than that of the ground states.

If $p > 1 + \frac{4}{n-2}$ then $s_p > 1$ and we are in the energy supercritical regime. Since for all $\epsilon > 0$ there exists $c_\epsilon > 0$ such that $\left| |u|^{\frac{4}{n-2}} u \right| \lesssim \left| |u|^{\frac{4}{n-2}} u g(|u|) \right| \leq c_\epsilon \max(1, \|u\|_{\frac{4}{n-2}+\epsilon}^{\frac{4}{n-2}+\epsilon})$ then the nonlinearity of (1) is said to be barely supercritical.

In this paper we study the asymptotic behavior of \tilde{H}^k -solutions of (1) for $n \in \{3, 4\}$. Recall the local-wellposed result:

Proposition 1. “Local well-posedness ” [11] *Let $n \in \{3, 4\}$ and let $k > \frac{n}{2}$. Let M be such that $\|u_0\|_{\tilde{H}^k} \leq M$. Then there exists $\epsilon := \epsilon(M) > 0$ small such that if $T_l > 0$ ($T_l :=$ time of local existence) satisfies*

$$(5) \quad \|e^{it\Delta} u_0\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}([0, T_l])} \leq \epsilon$$

then there exists a unique

$$(6) \quad u \in \mathcal{C}([0, T_l], \tilde{H}^k) \cap L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}([0, T_l]) \cap D^{-1} L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}}([0, T_l]) \\ \cap L_t^{\frac{2(n+2)}{n}} D^{-k} L_x^{\frac{2(n+2)}{n}}([0, T_l])$$

such that

$$(7) \quad u(t) = e^{it\Delta} u_0 + i \int_0^t e^{i(t-t')\Delta} \left(|u(t')|^{\frac{4}{n-2}} u(t') g(|u(t')|) \right) dt'$$

is satisfied in the sense of distributions. Here $D^{-\alpha} L^r := \dot{H}^{\alpha, r}$ endowed with the norm $\|f\|_{D^{-\alpha} L^r} := \|D^\alpha f\|_{L^r}$.

This allows to define the notion of maximal time interval of existence $I_{max} := (T_-, T_+)$, that is the union of all the intervals I containing 0 such that (7) holds in the class $\mathcal{C}(I, \tilde{H}^k) \cap L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(I) \cap D^{-1} L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}}(I) \cap D^{-k} L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}}(I)$. Recall the following proposition:

Proposition 2. “Global well-posedness: criterion” [11] *If $|I_{max}| < \infty$ then*

$$(8) \quad \|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(I_{max})} = \infty$$

Remark 1. *Proposition 1 and 2 were proved in [11] for solutions of barely supercritical defocusing nonlinearities. The proof can be easily adapted to solutions of (1).*

With this in mind, global well-posedness follows from an *a priori* bound of the form $\|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}([-T, T])} \leq f(T, \|u_0\|_{\tilde{H}^k})$ for arbitrarily large time $T > 0$. In fact for some data we shall prove that the bound does not depend on time T , which will show scattering.

Before stating the main theorem, we recall some general notation. We write $a \ll b$ if the value of a is much smaller than that of b , $a \gg b$ if the value of a is much larger than that of b , and $a \sim b$ if $a \ll b$ and $b \ll a$ are not true. We say that \tilde{C} is the constant determined by $a \lesssim b$ if it is the smallest constant C such that $a \leq Cb$. We write $a = o(b)$ if there exists a constant $0 < c \ll 1$ such that $|a| \leq c|b|$. We define $b+ = b + \epsilon$ for $0 < \epsilon \ll 1$. If $b+$ appears in a mathematical expression such as $a \leq Cb+$, then we ignore the dependence of C on ϵ in order to make our presentation simple.

Let $2^* := \frac{2n}{n-2}$. If $f \in \tilde{H}^k$ then we define the energy

$$(9) \quad E(f) := \frac{1}{2} \int_{\mathbb{R}^n} |\nabla f(x)|^2 - \int_{\mathbb{R}^n} F(f, \bar{f})(x) dx,$$

with

$$(10) \quad F(z, \bar{z}) := \int_0^{|z|} s^{\frac{n+2}{n-2}} g(s) ds.$$

Indeed

$$(11) \quad \left| \int_{\mathbb{R}^n} F(f, \bar{f})(x) dx \right| \lesssim \|f\|_{L^{2^*}}^{2^*} g(\|f\|_{L^\infty}) \lesssim \|f\|_{\tilde{H}^1}^{2^*} g(\|f\|_{\tilde{H}^k}) :$$

this follows from a simple integration by part

$$(12) \quad F(z, \bar{z}) \sim |z|^{\frac{2n}{n-2}} g(|z|),$$

combined with the Sobolev inequality

$$(13) \quad \|u\|_{L_t^\infty L_x^\infty(J)} \lesssim \|u\|_{L_t^\infty \tilde{H}^k(J)}.$$

If $f \in \dot{H}^1$ then we denote by $\tilde{E}(f)$ the following

$$(14) \quad \tilde{E}(f) := \frac{1}{2} \int_{\mathbb{R}^n} |\nabla f(x)|^2 - \frac{1}{2^*} \int_{\mathbb{R}^n} |f(x)|^{2^*} dx.$$

Hence $\tilde{E}(f) = E(f) + X(f)$ with

$$X(f) := \int_{\mathbb{R}^n} \int_0^{|f(x)|} (g(s) - 1) s^{2^*-1} ds dx.$$

We define the functional

$$\tilde{K}(f) := \|\nabla f\|_{L^2}^2 - \|f\|_{L^{2^*}}^{2^*}.$$

Let u be an \tilde{H}^k solution of (1). A simple computation shows that the energy $E(u(t))$ is conserved, or, in other words, that $E(u(t)) = E(u_0)$. Let χ be a smooth,

radial function supported on $|x| \leq 2$ such that $\chi(x) = 1$ if $|x| \leq 1$. If $x_0 \in \mathbb{R}^n$, $R > 0$, then we define the mass within the ball $B(x_0, R)$

$$(15) \quad \text{Mass}(B(x_0, R), u(t)) := \left(\int_{B(x_0, R)} |u(t, x)|^2 dx \right)^{\frac{1}{2}}$$

Recall (see [13]) that

$$(16) \quad \text{Mass}(B(x_0, R), u(t)) \lesssim R \sup_{t' \in [0, t]} \|\nabla u(t')\|_{L^2}$$

and that its derivative satisfies

$$(17) \quad |\partial_t \text{Mass}(u(t), B(x_0, R))| \lesssim \frac{\sup_{t' \in [0, t]} \|\nabla u(t')\|_{L^2}}{R}$$

The main result of this paper is:

Theorem 3. *Let $\delta > 0$, $n \in \{3, 4\}$, and $u_0 \in \tilde{H}^k := \dot{H}^k(\mathbb{R}^n) \cap \dot{H}^1(\mathbb{R}^n)$, $k > \frac{n}{2}$, radial such that¹*

$$(18) \quad \|u_0\|_{\tilde{H}^k} \gtrsim 1, \quad E(u_0) < (1 - 2\delta)\tilde{E}(W), \quad \text{and} \quad \|u_0\|_{L^{2^*}} < \|W\|_{L^{2^*}}.$$

Let $a_n := 365$ (resp. $a_n := \frac{683}{6}$) if $n = 3$ (resp. $n = 4$). There exist $C_a \gg 1$ such that if $\gamma > 0$ satisfies

$$(19) \quad \log^\gamma \log \log \left(C_a^{C_a} \delta^{\frac{1}{2(\alpha-1)+}} \|u_0\|_{\tilde{H}^k}^2 \right) - 1 \ll \delta$$

with $\alpha := \frac{1}{\gamma a_n}$ then the solution of (1) with data $u(0) := u_0$ exists for all time T . Moreover there exists a scattering state $u_{0,+} \in \tilde{H}^k$ such that

$$(20) \quad \lim_{t \rightarrow \infty} \|u(t) - e^{it\Delta} u_{0,+}\|_{\tilde{H}^k} = 0$$

and there exists $C := C(\|u_0\|_{\tilde{H}^k}, \delta)$ such that $\|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(\mathbb{R})} \leq C$.

By symmetry an analogous statement holds for negative times.

Remark 2. *Notice that $\|u_0\|_{L^{2^*}} = \|W\|_{L^{2^*}}$ is impossible. Indeed, recall [1, 10] the sharp Sobolev inequality $\|f\|_{L^{2^*}} \leq C_* \|\nabla f\|_{L^2}$ with $C_* := \frac{\|W\|_{L^{2^*}}}{\|\nabla W\|_{L^2}}$. Hence $F(y) < (1 - \delta)F(\|W\|_{L^{2^*}})$ with $y := \|u_0\|_{L^{2^*}}$ and*

$$(21) \quad F(y) := \frac{1}{2} C_*^2 y^2 - \frac{1}{2^*} y^{2^*}.$$

Remark 3. *Observe from (13) and (19) that $\|g(u_0) - 1\|_{L^\infty} \ll \delta$. Hence $\tilde{E}(u_0) < (1 - \delta)\tilde{E}(W)$. Hence $0 < \delta\tilde{E}(W) < \frac{1}{2} (\|\nabla W\|_{L^2}^2 - \|\nabla u_0\|_{L^2}^2)$. So in particular δ is bounded from above by a constant that does not depend on $\|u_0\|_{\tilde{H}^k}$.*

Remark 4. *For data satisfying (18), Theorem 3 implies global regularity since by the Sobolev embedding $\|u\|_{L_t^\infty L_x^\infty(\mathbb{R})} \lesssim \|u\|_{L_t^\infty \tilde{H}^k(\mathbb{R})}$ for $k > \frac{n}{2}$.*

¹If we only assume that $\|u_0\|_{\tilde{H}^k} \ll 1$ then the same conclusion holds. This is a consequence of the local theory: see Appendix.

We recall some standard inequalities. The following Sobolev inequality holds:

$$(22) \quad \|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J)} \lesssim \|Du\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}(J)}.$$

If u is a solution of $i\partial_t u + \triangle u = G$, $u(t=0) := u_0$ on J such that $u(t) \in \tilde{H}^k$, $t \in J$, then the Strichartz estimates (see for example [6]) yield

$$(23) \quad \begin{aligned} & \|u\|_{L_t^\infty \dot{H}^j(J)} + \|D^j u\|_{L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}}(J)} + \|D^j u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2n(n+2)}{n^2+4}}(J)} \\ & \lesssim \|D^j G\|_{L_t^{\frac{2(n+2)}{n+4}} L_x^{\frac{2(n+2)}{n+4}}(J)} + \|u_0\|_{\dot{H}^j} \end{aligned}$$

if $j \in \{1, k\}$. We define

$$(24) \quad Q(J, u) := \|u\|_{L_t^\infty \tilde{H}^k(J)} + \|Du\|_{L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}}(J)} + \|D^k u\|_{L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}}(J)} + \|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J)}.$$

We also recall the following proposition:

Proposition 4. “A fractional Leibnitz rule”[11] *Let $0 \leq \alpha \leq 1$, k and β integers such that $k \geq 2$ and $\beta \geq 1$, $(r, r_1, r_2) \in (1, \infty)^3$, $r_3 \in (1, \infty]$ be such that $\frac{1}{r} = \frac{\beta}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$. Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a C^k -function and let $G := \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ be a C^k -function such that*

$$F^{[i]}(x) = O\left(\frac{F(x)}{x^i}\right), \quad \tau \in [0, 1] : |F^{[j]}(\tau x + (1-\tau)y)| \lesssim |F^{[j]}(x)| + |F^{[j]}(y)|$$

and

$$(25) \quad |G^{[i]}(x, \bar{x})| = \begin{cases} O(|x|^{\beta+1-i}), & i \leq \beta + 1 \\ 0, & i > \beta + 1 \end{cases}$$

for $0 \leq i \leq k$ and $1 \leq j \leq k$. Then

$$(26) \quad \|D^{k-1+\alpha}(G(f, \bar{f})F(|f|))\|_{L^r} \lesssim \|f\|_{L^{r_1}}^\beta \|D^{k-1+\alpha} f\|_{L^{r_2}} \|F(|f|)\|_{L^{r_3}}$$

Here $F^{[i]}$ and $G^{[i]}$ denote the i^{th} -derivatives of F and G respectively.

Now we explain how this paper is organized. In Section 2 we prove the main result of this paper, i.e Theorem 3. The proof relies upon the following bound of $\|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}}$ on an arbitrarily long time interval

Proposition 5. “Bound of $L_t^{\frac{2(n+2)}{2(n-2)}} L_x^{\frac{2(n+2)}{2(n-2)}}$ norm” *Let u be an \tilde{H}^k solution of (1) such that (18) holds. Let $J := [t_1, t_2]$ be an interval. Assume that for all $t \in J$*

$$(27) \quad g(u(t)) - 1 \ll \delta.$$

Then there exists a constant $C_0 \gg 1$ such that if $\|u\|_{L_t^\infty \tilde{H}^k(J)} \leq M$ for some $M \gg 1$, then

$$(28) \quad \|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J)} \leq C_0^{\delta^{-\frac{1}{2}} g^{a_n}(M)}$$

This bound proved on an arbitrary time interval J , combined with a local induction on time of some Strichartz estimates, allows to control *a posteriori* the $L_t^\infty \tilde{H}^k$ norm of the solution and some other norms at \tilde{H}^k regularity on J , and to show *a posteriori* that the condition (27) holds on J , assuming that g grows slowly enough, in the sense of (19): see [11, 12] for a similar argument. Global well-posedness and scattering of \tilde{H}^k -solutions of (1) follow easily from the finiteness of these bounds. In Section 3, we prove Proposition 5. We mention the main differences between this paper and [11]. First one has to assume the condition (27). This condition, combined with the energy conservation law and the variational properties of the ground states, assure that some relevant norms (such as the kinetic energy and the potential energy) are bounded on J , so that we can apply the techniques of concentration (see e.g [2, 13]) in order to prove (28). Roughly speaking, we divide J into subintervals $(J_l)_{1 \leq l \leq L}$ such that the $L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}$ norm of u concentrates, i.e it is small but also substantial. Our goal is to estimate the number of these subintervals. It is already known that the mass on a ball centered at the origin concentrates for all time of each of these subintervals. In [11], a Morawetz-type estimate (combined with the mass concentration) was used to prove that the following statement holds: one of these subintervals is large compare with J . In this paper we use the virial identity and we adapt arguments in [9, 14] to prove a decay at some time of the potential energy on a ball centered at the origin, which leads to a contradiction unless the statement above holds. With the use of this statement one can show that there exists a significant number of subintervals (in comparison with the total number of subintervals) that concentrate around some time and such that the mass concentrates around the origin, which yields an estimate of the number of all the subintervals. The process involves several estimates. One has to understand how they depend on $g(M)$ and δ since this will play an important role in the choice of γ (see (19)) for which we have global well-posedness and scattering of \tilde{H}^k -solutions of (1).

2. PROOF OF THEOREM 3

The proof is made of two steps:

- finite bound of $\|u\|_{L_t^\infty \tilde{H}^k(\mathbb{R})}$, $\|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(\mathbb{R})}$, $\|Du\|_{L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}}(\mathbb{R})}$ and $\|D^k u\|_{L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}}(\mathbb{R})}$. By time reversal symmetry² and by monotone convergence it is enough to find, for all $T \geq 0$, a finite bound of all these norms restricted to $[0, T]$ and the bound should not depend on T .

$$(29) \quad \mathcal{F} := \left\{ T \in [0, \infty) : \sup_{t \in [0, T]} Q([0, t], u) \leq M_0; g(u(t)) - 1 \ll \delta \right\}$$

We claim that $\mathcal{F} = [0, \infty)$ for M_0 , a large constant (to be chosen later) depending only on $\|u_0\|_{\tilde{H}^k}$ and δ . Indeed

- $0 \in \mathcal{F}$.
- \mathcal{F} is closed by continuity

²i.e if $t \rightarrow u(t, x)$ is a solution of (1) then $t \rightarrow \bar{u}(-t, x)$ is also a solution of (1)

– \mathcal{F} is open. Indeed let $T \in \mathcal{F}$. Then, by continuity there exists $\delta' > 0$ such that for $T' \in [0, T + \delta']$ we have $Q([0, T']) \leq 2M_0$. In view of (28), this implies, in particular, that

$$(30) \quad \|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}([0, T'])} \leq C_0^{\delta - \frac{1}{2} g^{an + (2M_0)}}$$

Let $J := [0, a]$ be an interval. We get from (23), (13), and Proposition 4

$$(31) \quad \begin{aligned} Q(J, u) &\lesssim \|u_0\|_{\tilde{H}^k} + \left(\|Du\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J)} + \|D^k u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J)} \right) \|u\|_{L_t^{\frac{4}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J)} \\ &\quad g \left(\|u\|_{L_t^\infty \tilde{H}^k(J)} \right) \\ &\lesssim \|u_0\|_{\tilde{H}^k} + Q(J, u) \|u\|_{L_t^{\frac{4}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J)} g(Q(J, u)) \end{aligned}$$

Let C be the constant determined by \lesssim in the second line of (31). We may assume WLOG that $C \gg 1$. Let $0 < \epsilon \ll 1$. Notice that if J satisfies $\|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J)} = \frac{\epsilon}{g^{\frac{n-2}{4}} (2C\|u_0\|_{\tilde{H}^k})}$ then a simple continuity argument shows that

$$(32) \quad Q(J, u) \leq 2C\|u_0\|_{\tilde{H}^k}.$$

We divide $[0, T']$ into subintervals $(J_i)_{1 \leq i \leq I}$ such that $\|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J_i)} = \frac{\epsilon}{g^{\frac{n-2}{4}} ((2C)^i \|u_0\|_{\tilde{H}^k})}$, $1 \leq i < I$ and $\|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J_I)} \leq \frac{\epsilon}{g^{\frac{n-2}{4}} ((2C)^I \|u_0\|_{\tilde{H}^k})}$. We will see shortly that such a partition exists. First we prove the following claim:

Claim: Let $X := \sum_{i=1}^{I-1} \frac{1}{g^{\frac{n+2}{2}} ((2C)^i \|u_0\|_{\tilde{H}^k})}$.

There exists $\bar{C} \gg 1$ such that if $\bar{i} := [\bar{C}\|u_0\|_{\tilde{H}^k}]^3$, then

$$(33) \quad X \gtrsim \frac{I-1-\bar{i}}{\log \frac{(n+2)\gamma}{2}(\bar{i})} + \frac{\bar{i}}{\log \frac{(n+2)\gamma}{2} \log(\bar{i})}$$

Proof. We get $g^{\frac{n+2}{2}} ((2C)^i \|u_0\|_{\tilde{H}^k}) \approx \log^{\frac{(n+2)\gamma}{2}} (\log(\log((2C)^{2i} \|u_0\|_{\tilde{H}^k}^2)))$. Hence choosing $\bar{C} \gg 1$ we see that

$$\begin{aligned} X &\gtrsim \sum_{i=1}^{\bar{i}} \frac{1}{\log \frac{(n+2)\gamma}{2} \log(\bar{i})} + \sum_{i=\bar{i}+1}^{I-1} \frac{1}{\log \frac{(n+2)\gamma}{2} \log(i)} \\ &= X_1 + X_2. \end{aligned}$$

³Here $[x]$ denotes the entire part of x

Let $F(I) := \int_{\bar{i}}^{I-1} \frac{1}{\log \frac{(n+2)\gamma}{2} \log(x)} dx$. Observe that $X_2 \geq F(I) \gtrsim \frac{I-1-\bar{i}}{\log \frac{(n+2)\gamma}{2} \log(I)}$, where the last estimate can be deduced easily from integrating $F(I)$ once by parts. Hence (33) holds. \square

We have

$$(34) \quad C_0^{C_0^{\delta-\frac{1}{2}} g^{a_n^+}(2M_0)} \gtrsim X \gtrsim \frac{I-1-\bar{i}}{\log \frac{(n+2)\gamma}{2} \log(I)} + \frac{\bar{i}}{\log \frac{(n+2)\gamma}{2} \log(\bar{i})}$$

Moreover, by iterating the procedure in (31) and (32) we get

$$(35) \quad Q([0, T'], u) \leq (2C)^I \|u_0\|_{\tilde{H}^k}.$$

In view of (35), we may assume WLOG that $I \gg \bar{i}$. Hence we see from (34) that

$$\log \log(I) \lesssim \delta^{-\frac{1}{2}} g^{a_n^+}(2M_0).$$

Since $\gamma \ll 1$ we see that we can choose $M_0 := M_0(\|u_0\|_{\tilde{H}^k}, \delta)$ such that

$$(36) \quad \delta^{-\frac{1}{2}} g^{a_n^+}(2M_0) \ll \log \log \left(\frac{\log \left(\frac{M_0}{\|u_0\|_{\tilde{H}^k}} \right)}{\log(2C)} \right).$$

This shows that $Q([0, T'], u) \leq M_0$. We now find values of M_0 for which (36) holds. Let $M'_0 := \frac{M_0}{\|u_0\|_{\tilde{H}^k}}$. Choose $M_0 \gg \|u_0\|_{\tilde{H}^k}$. Hence, with this choice of M_0 , it is sufficient to find M'_0 such that

$$\frac{\log^{\alpha-} \log \log(M'_0)}{\log \log \log(M'_0)} \gg \frac{1}{\delta^{\frac{\alpha}{2}-}}$$

which is satisfied if

$$M'_0 \geq \bar{C}^{\bar{C}^{\bar{C}}} \delta^{\frac{1}{2(\alpha-1)^+}}$$

for some well-chosen constant $\bar{C} \gg 1$. Hence we see from (19) that $g(u(t)) - 1 \ll \delta$ for $t \in [0, T']$.

- Scattering: it is enough to prove that $e^{-it\Delta}u(t)$ has a limit as $t \rightarrow \infty$ in \tilde{H}^k . If $t_1 < t_2$ then by dualizing (23) with $G = 0$ we get

$$(37) \quad \begin{aligned} & \|e^{-it_1\Delta}u(t_1) - e^{-it_2\Delta}u(t_2)\|_{\tilde{H}^k} \\ & \lesssim \|u\|_{L_t^{\frac{4}{n-2}} L_x^{\frac{2(n+2)}{n-2}}([t_1, t_2])} \left(\|D^k u\|_{L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}}([t_1, t_2])} + \|Du\|_{L_t^{\frac{2(n+2)}{n}} L_x^{\frac{2(n+2)}{n}}([t_1, t_2])} \right) g(M_0), \end{aligned}$$

and we conclude that there exists $A(\epsilon)$ such that if $t_2 \geq t_1 \geq A(\epsilon)$ then $\|e^{-it_1\Delta}u(t_1) - e^{-it_2\Delta}u(t_2)\|_{\tilde{H}^k} \leq \epsilon$. Hence scattering.

3. PROOF OF PROPOSITION 5

In this section we prove Proposition 5. First we prove a preliminary lemma.

3.1. A lemma.

Lemma 6. *There exists $\delta' \approx \delta^{\frac{1}{2}}$ such that for all $t \in J$*

$$(38) \quad \tilde{E}(u(t)) < (1 - \delta) \tilde{E}(W)$$

and

$$(39) \quad \begin{aligned} \int_{\mathbb{R}^n} |\nabla u(t)|^2 dx &\leq (1 - \delta') \int_{\mathbb{R}^n} |\nabla W|^2 dx \\ \tilde{K}(u(t)) &\geq \delta' \int_{\mathbb{R}^n} |\nabla u(t)|^2 dx \end{aligned}$$

Proof. Recall that $\tilde{E}(W) := (\frac{1}{2} - \frac{1}{2^*}) \|W\|_{L^{2^*}}^{2^*}$. Define

$$\mathcal{F} := \{T \in J : (39) \text{ holds for } t \in [t_1, T]\}$$

We claim that $\mathcal{F} = J$. Clearly $t_1 \in \mathcal{F}$ and \mathcal{F} is closed by continuity of the flow. It remains to prove that it is open. By continuity there exists $\beta > 0$ such that (39) holds for $T' \in [t_1, T + \beta]$ with δ' substituted for $2\delta'$. Hence in view of the assumptions, the Sobolev embedding, and the conservation of the energy

$$(40) \quad \tilde{E}(u(t)) = E(u) + X(u(t)) < (1 - 2\delta) \tilde{E}(W) + \delta < (1 - \delta) \tilde{E}(W)$$

Let us define

$$\begin{aligned} (a) : & \|u(t)\|_{L^{2^*}} < \|W\|_{L^{2^*}} \\ (b) : & \|\nabla u(t)\|_{L^2} < \|\nabla W\|_{L^2} \end{aligned}$$

With (40) in mind it is left to the reader to check that if t satisfies (a) then t satisfies (b). Recall [7] that

$$\|\nabla u(t)\|_{L^2}^2 < \|\nabla W\|_{L^2}^2 \Rightarrow (39) \text{ holds}$$

Hence $T' \in \mathcal{F}$. □

3.2. The proof. We prove now Proposition 5 by using this lemma and concentration techniques (see e.g [2, 13]).

We divide the interval $J = [t_1, t_2]$ into subintervals $(J_l := [\bar{t}_l, \bar{t}_{l+1}])_{1 \leq l \leq L}$ such that

$$(41) \quad \|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J_l)} = \eta_1$$

$$(42) \quad \|u\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J_L)} \leq \eta_1$$

with $c_1 \ll 1$ and $\eta_1 := \frac{c_1}{g^{\frac{2(n+2)}{6-n}}(M)}$. In view of (28), we may replace WLOG the “ \leq ” sign with the “ $=$ ” sign in (42).

Recall the notion of exceptional intervals and the notion of unexceptional intervals (such a notion appears in the study of (3) in [13]). Let

$$(43) \quad \eta_2 := \begin{cases} c_2 (\eta_1 g^{-1}(M))^{22}, & n = 3 \\ c_2 (\eta_1^{35} g^{-28}(M))^{\frac{1}{3}}, & n = 4 \end{cases}$$

with $c_2 \ll c_1$. An interval $J_{l_0} = [\bar{t}_{l_0}, \bar{t}_{l_0+1}]$ of the partition $(J_l)_{1 \leq l \leq L}$ is exceptional if

$$(44) \quad \|u_{l,t_1}\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J_{l_0})} + \|u_{l,t_2}\|_{L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J_{l_0})} \geq \eta_2.$$

In view of (22) and (23), we have

$$(45) \quad \text{card} \{J_l : J_l \text{ exceptional}\} \lesssim \eta_2^{-1}.$$

Recall that on each unexceptional subintervals J_l there is a ball for which we have a mass concentration.

Result 1. “Mass concentration” [11] *There exists an $x_l \in \mathbb{R}^n$, two constants $0 < c \ll 1$ and $C \gg 1$ such that for each unexceptional interval J_l and for $t \in J_l$*

- if $n = 3$

$$(46) \quad \text{Mass} \left(u(t), B(x_l, Cg^{\frac{13}{3}}(M)|J_l|^{\frac{1}{2}}) \right) \geq cg^{-\frac{13}{3}}(M)|J_l|^{\frac{1}{2}}$$

- if $n = 4$

$$(47) \quad \text{Mass} \left(u(t), B(x_l, Cg^{\frac{17}{3}}(M)|J_l|^{\frac{1}{2}}) \right) \geq cg^{-\frac{17}{3}}(M)|J_l|^{\frac{1}{2}}$$

The radial symmetry allows to prove that, in fact, there is a mass concentration around the origin on each unexceptional interval J_l . Recall

Result 2. “Mass concentration around the origin”[11] *There exist a positive constant $\ll 1$ (that we still denote by c to avoid too much notation) and a constant $\tilde{C} \gg 1$ such that on each unexceptional interval J_l we have*

- if $n = 3$

$$(48) \quad \text{Mass} \left(u(t), B(0, \tilde{C}g^{\frac{169}{3}}(M)|J_l|^{\frac{1}{2}}) \right) \geq cg^{-\frac{13}{3}}(M)|J_l|^{\frac{1}{2}}$$

- if $n = 4$

$$(49) \quad \text{Mass} \left(u(t), B(0, \tilde{C}g^{51}(M)|J_l|^{\frac{1}{2}}) \right) \geq cg^{-\frac{17}{3}}(M)|J_l|^{\frac{1}{2}}$$

Remark 5. Notice that the values of the parameters η_1 and η_2 are not chosen randomly. They are the largest ones such that all the constraints appearing in the proof of Result 1 and Result 2 are satisfied.

Let $\tilde{J} := J_{i_0} \cup \dots \cup J_{i_1}$ be a sequence of consecutive unexceptional intervals. Let $L_{\tilde{J}}$ be the number of unexceptional intervals making \tilde{J} . Observe that

$$(50) \quad \text{Number of } \tilde{J}\text{'s} \lesssim \eta_2^{-1}.$$

We claim that one of the intervals $J_l \in \tilde{J}$ is large. More precisely

Result 3. “One of the intervals J_l is large ” *There exists a positive constant $\ll 1$ (that we still denote by c to avoid too much notation) and $\tilde{l} \in [i_0, \dots, i_1]$ such that*

- if $n = 3$

$$(51) \quad |J_{\tilde{l}}| \geq c^{\delta - \frac{1}{2}} g^{365+}(M) |\tilde{J}|$$

- if $n = 4$

$$(52) \quad |J_{\tilde{l}}| \geq c^{\delta - \frac{1}{2}} g^{\frac{683}{6}+}(M) |\tilde{J}|$$

Proof. Let a be a smooth function. Let u be a solution of $i\partial_t u + \Delta u = G$. Let $\{G, f\}_p := \Re(G\overline{\nabla f} - f\overline{\nabla G})$. Recall the following facts (see e.g [4]):

- if G is of the form $G(z) := F'(|z|^2)z$ then $\{G, f\}_p = -\nabla H(|f|^2)$ with $H(x) := xF'(x) - F(x)$.
-

$$(53) \quad \partial_t M_a = \int (-\Delta \Delta a) |u|^2 + 4\partial_{x_j x_k}^2 a \Re(\overline{\partial_{x_j} u} \partial_{x_k} u) + 2\partial_{x_j} a \{G, u\}_p^j dx,$$

with $M_a(t) := \int 2\partial_{x_j} a \Im(\bar{u} \partial_{x_j} u) dx$.

Let $m \in \mathbb{R}^+$ and $G(z) := -|z|^{\frac{4}{n-2}} z g(|z|)$. Let $a(x) := m^2 \phi\left(\frac{|x|}{m}\right)$ with ϕ a smooth, radial, and compactly supported function such that $\phi(x) = |x|^2$ for $|x| \leq 1$. Then

$$2m \partial_t \left\langle \phi' \left(\frac{|x|}{m} \right) \frac{x_j}{|x|} u, -i \partial_{x_j} u \right\rangle = \int (-\Delta \Delta a) |u|^2 + 4\partial_{x_j x_k}^2 a \Re(\overline{\partial_{x_j} u} \partial_{x_k} u) - 2\partial_{x_j} a \partial_{x_j} H(|u|^2) dx,$$

with

$$H(x) := -x^{\frac{n}{n-2}} \tilde{g}(x) + \int_0^x s^{\frac{2}{n-2}} \tilde{g}(s) ds, \quad \tilde{g}(x) := \log^\gamma \log \log (10^{10} + x).$$

Integrating by parts once the second term of $H(x)$ we see that if $x \geq 0$ then

$$(54) \quad H(x) = \left(\frac{n-2}{n} - 1 \right) x^{\frac{n}{n-2}} \tilde{g}(x) - \frac{n-2}{n} \int_0^x s^{\frac{n}{n-2}} \tilde{g}'(s) ds, \text{ and } H(x) \sim -x^{\frac{n}{n-2}} \tilde{g}(x).$$

Hence one can write

$$\begin{aligned} 2m \partial_t \left\langle \phi' \left(\frac{|x|}{m} \right) \frac{x_j}{|x|} u, -i \partial_{x_j} u \right\rangle &= 8 \int_{|x| \leq m} |\nabla u(t)|^2 - |u(t)|^{2^*} dx - 8 \int_{|x| \leq m} (g(u(t)) - 1) |u|^{2^*} dx \\ &\quad + O(X_m) + O(Y_m) \\ &\geq 8\tilde{K}(\chi_m u(t)) - 8 \sup_{t \in \tilde{J}} (g(u(t)) - 1) \|u\|_{L^{2^*}(|x| \leq m)}^{2^*} + O(X_m) + O(Y_m). \end{aligned}$$

Here $\chi_m(x) := \chi\left(\frac{|x|}{m}\right)$ with χ a smooth function compactly supported on $B(0, 2)$ such that $\chi(x) = 1$ if $|x| \leq 1$, $X_m := \int_{|x| \leq 2m} \int_0^{|u|^2} t^{\frac{n}{n-2}} \tilde{g}'(t) dt dx$, and $Y_m := \int_{|x| \geq m} \frac{m}{|x|} \left(|\partial_r u|^2 + \frac{|u|^2}{r^2} + |u|^{2^*} g(M) \right) dx$. We claim that there exists a constant $c' \approx \delta^{\frac{1}{2}}$ such that

$$(55) \quad \tilde{K}(\chi_m u) \geq c' \|\nabla(\chi_m u)\|_{L^2}^2.$$

Indeed (see [9] for a similar argument, see also [8] for the subcritical case) we see from the sharp Sobolev inequality, (40), and (39) in Lemma 6 that

$$\|\chi_m u\|_{L^{2^*}}^{2^*} < \|u\|_{L^{2^*}}^{2^*} < (1 - \delta') \|W\|_{L^{2^*}}^{2^*};$$

Observe also that since $\gamma \ll \delta^{\frac{1}{2}}$, we have

$$X_m = o\left(\delta^{\frac{1}{2}} \int_{|x| \leq m} |u(t)|^{2^*} dx\right)$$

Hence we see that there exists $c' \approx \delta^{\frac{1}{2}}$ such that

$$\begin{aligned} \tilde{K}(\chi_m u) &\geq \|\nabla(\chi_m u)\|_{L^2}^2 \left(1 - C_*^2 \|\chi_m u\|_{L^{2^*}}^{2^*-2}\right) \\ &\geq \|\nabla(\chi_m u)\|_{L^2}^2 \left(1 - C_*^2 (1 - \delta')^{\frac{2^*-2}{2^*}} \|W\|_{L^{2^*}}^{2^*-2}\right) \\ &\geq c' \|\nabla(\chi_m u)\|_{L^2}^2. \end{aligned}$$

Hence, in view of (27) and (39) we have

$$\int_J \int_{|x| \leq m} |u(t)|^{2^*} dx dt \lesssim \delta^{-\frac{1}{2}} (m^2 + O(\int_J Y_m dt))$$

Hence, letting $\bar{m} := 2^{-\bar{K}} m$ for some $\bar{K} > 0$, we have (see [14] for a similar argument)

$$\sum_{m'=\bar{m}}^m \int_J \int_{|x| \leq m'} |u(t)|^{2^*} dx dt \lesssim \delta^{-\frac{1}{2}} (m^2 + |J|g(M)),$$

from which we get

$$\int_J \int_{|x| \leq \bar{m}} |u(t)|^{2^*} dx dt \lesssim \delta^{-\frac{1}{2}} (\bar{K}^{-1} m^2 + \bar{K}^{-1} |J|g(M)).$$

Hence there exists $\bar{t} \in J$ such that

$$\int_{|x| \leq 2^{-\bar{K}} m} |u(\bar{t})|^{2^*} dx \lesssim \delta^{-\frac{1}{2}} ((\bar{K}|J|)^{-1} m^2 + \bar{K}^{-1} g(M)).$$

Letting $m := g^{\frac{1}{2}}(M) |J|^{\frac{1}{2}}$ we get

$$(56) \quad \int_{|x| \leq 2^{-\bar{K}} g^{\frac{1}{2}}(M) |J|^{\frac{1}{2}}} |u(\bar{t})|^{2^*} dx \lesssim \delta^{-\frac{1}{2}} \bar{K}^{-1} g(M).$$

This implies that (52) holds with $\bar{K} := C'' \delta^{-\frac{1}{2}} g^{\frac{683}{3}}(M)$ for some fixed and well-chosen constant $C'' \gg 1$. If not we see from (49) and Hölder that this it violates (56). \square

Next we recall a crucial algorithm due to Bourgain [2] to prove that there are many of those intervals that concentrate.

Result 4. *Let*

$$(57) \quad \eta := \begin{cases} c^{\delta - \frac{1}{2}} g^{365+}(M), & n = 3 \\ c^{\delta - \frac{1}{2}} g^{\frac{683}{6}+}(M), & n = 4 \end{cases}$$

There exist a time \bar{t} , $K > 0$ and intervals J_{i_1}, \dots, J_{i_K} such that

$$(58) \quad |J_{i_1}| \geq 2|J_{i_2}| \dots \geq 2^{k-1}|J_{i_k}| \dots \geq 2^{K-1}|J_{i_K}|$$

such that

$$(59) \quad \text{dist}(\bar{t}, J_{i_k}) \leq \eta^{-1}|J_{i_k}|$$

and

$$(60) \quad K \geq -\frac{\log(L_{\bar{t}})}{2 \log(\frac{\eta}{8})}$$

A proof of this result in such a state can be found in [11]. With this result in mind we prove that $L_{\bar{t}} < \infty$. More precisely

Result 5. *There exists one constant $C_1 \gg 1$ such that*

- *if $n = 3$*

$$(61) \quad L_{\bar{t}} \leq C_1^{c^{\delta - \frac{1}{2}} g^{365+}(M)}$$

- *if $n = 4$*

$$(62) \quad L_{\bar{t}} \leq C_1^{c^{\delta - \frac{1}{2}} g^{\frac{683}{6}+}(M)}$$

Proof. We shall prove this result for $n = 4$. The case $n = 3$ is left to the reader. Let $C \gg 1$ be a constant that is allowed to change from one line to the other one and such that all the estimates below are true. Let $R_{i_{l_k}} := C^{\delta - \frac{1}{2}} g^{\frac{683}{6}+}(M) |J_{i_k}|^{\frac{1}{2}}$. By Result 1 we have

$$(63) \quad \text{Mass} \left(u(t), B(x_{i_{l_k}}, R_{i_{l_k}}) \right) \geq c g^{-\frac{17}{3}}(M) |J_{i_k}|^{\frac{1}{2}}$$

for all $t \in J_{i_k}$. By (17) and (59) we see that (63) holds for $t = \bar{t}$ with c substituted for $\frac{c}{2}$. On the other hand we see that by (16) and (58) that ⁴

$$\begin{aligned} \sum_{k'=k+N}^K \int_{B(x_{i_{l_{k'}}, R_{i_{l_{k'}}})} |u(\bar{t}, x)|^2 dx &\leq \left(\frac{1}{2^N} + \frac{1}{2^{N+1}} \dots + \frac{1}{2^{K-k}} \right) R_{i_{l_k}}^2 \\ &\leq \frac{1}{2^{N-1}} R_{i_{l_k}}^2. \end{aligned}$$

Now we let $N = C^{\delta - \frac{1}{2}} g^{\frac{683}{6}+}(M)$ so that $\frac{R_{i_{l_k}}^2}{2^{N-1}} \leq \frac{1}{8} c^2 g^{-\frac{34}{3}}(M) |J_{i_k}|$. By (63) we have

⁴Notation: $\sum_{k'=k+N}^K a_{k'} = 0$, if $k' > K$

$$\sum_{k'=k+N}^K \int_{B(x_{i_{l_{k'}}}, R_{i_{l_{k'}}})} |u(\bar{t}, x)|^2 dx \leq \frac{1}{2} \int_{B(x_{i_{l_k}}, R_{i_{l_k}})} |u(\bar{t}, x)|^2 dx.$$

Therefore

$$\begin{aligned} \int_{B(x_{i_{l_k}}, R_{i_{l_k}}) \cup \bigcup_{k'=k+N}^K B(x_{i_{l_{k'}}}, R_{i_{l_{k'}}})} |u(\bar{t}, x)|^2 dx &\geq \frac{1}{2} \int_{B(x_{i_{l_k}}, R_{i_{l_k}})} |u(\bar{t}, x)|^2 dx \\ &\geq \frac{c^2 g^{-\frac{34}{3}}(M)}{4} |J_{i_{l_k}}|, \end{aligned}$$

and by Hölder inequality, there exists a small constant (that we still denote by c) such that

$$\int_{B(x_{i_{l_k}}, R_{i_{l_k}}) \cup \bigcup_{k'=k+N}^K B(x_{i_{l_{k'}}}, R_{i_{l_{k'}}})} |u(\bar{t}, x)|^{\frac{2n}{n-2}} dx \geq c^{\delta - \frac{1}{2} g^{\frac{683}{6}}(M)},$$

and after summation over k , we get

$$\frac{K}{N} c^{\delta - \frac{1}{2} g^{\frac{683}{6}}(M)} \lesssim 1,$$

from $\sum_{k=1}^K \chi_{B(x_{i_{l_k}}, R_{i_{l_k}}) \cup \bigcup_{k'=k+N}^K B(x_{i_{l_{k'}}}, R_{i_{l_{k'}}})} \leq N$, the Sobolev inequality, and (39).

Hence from (60) we see that there exists a constant $C_1 \gg 1$ such that (62) holds. \square

In view of Result 5, (41), (45) and (50), we see that (28) holds.

APPENDIX

Proposition 4 and the Strichartz estimates allow to show that if $\|u_0\|_{\tilde{H}^k} \ll 1$, then the solution u constructed by Proposition 1 satisfies (20). Indeed (5) is satisfied on \mathbb{R} and for all J interval of \mathbb{R}

$$\begin{aligned} Q(J, u) &\lesssim \|u_0\|_{\tilde{H}^k} + \|D(|u|^{\frac{4}{n-2}} u g(|u|))\|_{L_t^{\frac{2(n+2)}{n+4}}(J) L_x^{\frac{2(n+2)}{n+4}}} + \|D^k(|u|^{\frac{4}{n-2}} u g(|u|))\|_{L_t^{\frac{2(n+2)}{n+4}}(J) L_x^{\frac{2(n+2)}{n+4}}} \\ &\lesssim \max \left(\|Du\|_{L_t^{\frac{2(n+2)}{n}}(J) L_x^{\frac{2(n+2)}{n}}}, \|D^k u\|_{L_t^{\frac{2(n+2)}{n}}(J) L_x^{\frac{2(n+2)}{n}}} \right) \|u\|_{L_t^{\frac{4}{n-2}} L_x^{\frac{2(n+2)}{n-2}}(J)}^{\frac{4}{n-2}} g(\|u\|_{L_t^\infty(J) \tilde{H}^k}) \\ &\lesssim \|u_0\|_{\tilde{H}^k} + Q^{\frac{n+2}{n-2}}(J) g(Q(J, u)) \end{aligned}$$

Hence a continuity argument shows that $Q(\mathbb{R}, u) < \infty$.

By dualizing (23) with $G = 0$ (more precisely the estimate $\|D^j u\|_{L_t^{\frac{2(n+2)}{n}}([t_1, t_2]) L_x^{\frac{2(n+2)}{n}}} \lesssim \|u_0\|_{\tilde{H}^j}$) we get

$$\begin{aligned} &\|e^{-it_1 \Delta} u(t_1) - e^{-it_2 \Delta} u(t_2)\|_{\tilde{H}^k} \\ &\lesssim \max \left(\|D(|u|^{\frac{4}{n-2}} u g(|u|))\|_{L_t^{\frac{2(n+2)}{n+4}}([t_1, t_2]) L_x^{\frac{2(n+2)}{n+4}}}, \|D^k(|u|^{\frac{4}{n-2}} u g(|u|))\|_{L_t^{\frac{2(n+2)}{n+4}}([t_1, t_2]) L_x^{\frac{2(n+2)}{n+4}}} \right) \\ &\lesssim \max \left(\|Du\|_{L_t^{\frac{2(n+2)}{n}}([t_1, t_2]) L_x^{\frac{2(n+2)}{n}}}, \|D^k u\|_{L_t^{\frac{2(n+2)}{n}}([t_1, t_2]) L_x^{\frac{2(n+2)}{n}}} \right) \|u\|_{L_t^{\frac{4}{n-2}} L_x^{\frac{2(n+2)}{n-2}}([t_1, t_2])}^{\frac{4}{n-2}} g(\|u\|_{L_t^\infty \tilde{H}^k([t_1, t_2])}) \end{aligned}$$

and we conclude from the previous step that there exists $A(\epsilon)$ such that if $t_2 \geq t_1 \geq A(\epsilon) > 0$, then $\|e^{-it_1\Delta}u(t_1) - e^{-it_2\Delta}u(t_2)\|_{\dot{H}^k} \leq \epsilon$. Hence (20) holds.

REFERENCES

- [1] Aubin, *Equations différentielles nonlineaires et probleme de Yamabe concernant la courbure scalaire*, J. Math. Pures. Appl. (9), 55, 1976, 3, 269-296.
- [2] J. Bourgain, *Global well-posedness of defocusing 3D critical NLS in the radial case*, JAMS 12 (1999), 145-171.
- [3] T. Cazenave, F.B. Weissler, *Critical nonlinear Schrödinger equation*, Non. Anal. TMA, 14 (1990), 807-836.
- [4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Global well-posedness and scattering in the energy space for the critical nonlinear Schrödinger equation in \mathbb{R}^3* , Annals of Math. 167 (2007), 767-865.
- [5] T. Duyckaerts and F. Merle, *Dynamic of threshold solutions for energy-critical NLS*, Geom. Funct. Anal. 18 (2009), no. 6, 1787-1840.
- [6] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. Math. J. 120 (1998), 955-980.
- [7] C. E. Kenig and F. Merle, *Global well-posedness, scattering, and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math., 166 (2006), no. 3, 645-675.
- [8] K. Nakanishi and W. Schlag, *Global dynamics above the ground state energy for the focusing nonlinear Klein-Gordon equation*, J. Diff. Eq. 250 (2011), 2299-2333.
- [9] K. Nakanishi and T. Roy, *Global dynamics above the ground state for the energy-critical Schrödinger equation with radial data*, Communications on Pure and Applied Analysis, Volume 15, Issue 6, November 2016, 2023-2058.
- [10] Talenti, Giorgio, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl.,(4) 1976, 353-372.
- [11] T. Roy, *Scattering above energy norm of solutions of a loglog energy-supercritical Schrödinger equation with radial data*, Journal of Differential Equations, 250 (2011), no. 1, 292-319. Erratum, to appear.
- [12] T. Roy, *Global existence of smooth solutions to a 3D loglog energy-supercritical wave equation*, Analysis and PDE, Vol 2, Num 3, 261-280, 2009.
- [13] T. Tao, *Global well-posedness and scattering for the higher-dimensional energy-critical non-linear Schrödinger equation for radial data*, New York J. Math., 11, 2005, 57-80.
- [14] T. Tao, *The Kenig-Merle scattering result for the energy-critical focusing NLS*, unpublished note.

NAGOYA UNIVERSITY, DEPARTMENT OF MATHEMATICS
 E-mail address: `tristanroy@math.nagoya-u.ac.jp`